محاضرات المرحلة الثالثة الفيزياء العامة

Complex analysis

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2022

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Complex analysis

A complex number

Is a number with the formula $(a+ib)$ whe
a arepeesent the Real part and b
is the Imaginary part -

A complex spase C is a Space with R
addition and multiplication by formula:

$(a_1, b_1), (a_2, b_2) \in R$

Real number R حيث

Remark (1)

a : is called Real part $[R_e z]$
b : is called Imaginary part $[I_m z]$
if $a = 0$ then $z(0, b)$ is called pure
imaginary number -

Remark (2)

$\forall a \in R, (a, 0) \Rightarrow a + i0 = a \in R$
$\forall b \in R, (0, b) \Rightarrow 0 + ib = ib \in C$

IF $i = (0,1)$

$i^2 = (\overset{a_1}{0}, \overset{b_1}{1}) * (\overset{a_2}{0}, \overset{b_2}{1}) =$

$i^2 = (0*0 - 1*1 \; , \; 0*1 + 0*1) = (-1, 0)$

لذلك سمي الجزء i بعنصر الوحدة التخيلي والذي يفسر حل للعديد من المشاكل الحسابية المستعصية .

IF $x^2 + 1 = 0$

$x^2 = -1$

$x = \sqrt{-1} = i$

∴ ان وجود الجزء i في نظام الاعداد العقدية يمكننا توضيح للجذور التربيعية للاعداد الحقيقية السالبة

example :-

IF it is $\underline{\underline{a}}$ positive Real number then the square root of $\underline{\underline{-a}}$ is :-

$\mp i \sqrt{a}$

$(i\sqrt{a})^2 = i^2 (\sqrt{a})^2 = -1 a = -a$

$(-i\sqrt{a})^2 = i^2 (\sqrt{a})^2 = -1 a = -a$

Define complex number Z

Z : يعرف بانه الزوج المرتبا (a,b) للاعداد الحقيقية a, b اي ان (a,b) والخاضع لعمليتي الجمع والضرب كالاتي

$Z = (a_1, b_1) \qquad Z_2 = (a_2, b_2)$

$Z_1 + Z_2 = (a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$

$Z_1 * Z_2 = (a_1, b_1) * (a_2, b_2)$

$= (a_1 \cdot a_2 - b_1 b_2, \; a_1 \cdot b_2 + a_2 b_1)$

$Z <$ جبرية $a + ib$ / قطبية

الازواج المرتبة من نوع $(a,0)$, $(0,b)$

تكون اعداد حقيقية ونكتبها a فقط اي ان

$a = (a, 0)$

اما الازواج المرتبة من نوع $(0,b)$ نسميها اعداد خيالية صرفة

فعليه كل عدد عقدي $Z = (a, b)$

$Z = (a, b) = (a, 0) + (0, b)$

$Z = a + ib$ الصيغة العامة للعدد العقدي

i وحدة الاعداد الخيالية

Properties of complex numbers

1- الزوج المرتب $(0,0)$ هو عدد حقيقي لان $a=0,\ b=0$
ويكتب $z = a+bi = 0+0i$

$Z = a+ib$

$Z_1 = a_1 + ib_1$

$Z_2 = a_2 + ib_2$

2- اذا كانت $Z_1 = Z_2$ فان $a_1 = a_2,\ b_1 = b_2$

example :-

Find the two numbers that the following equation :-

$(1-i)(2a-b) + (2+3i)(a+b) = 11-i$

$2a - b - i2a + ib + 2a + 2b + i3a + 3ib = 11-i$

$4a + b + (ai + 4ib) = 11-i$

$4a + b + (a+4b)i = 11-i$

$4a + b = 11 \quad * 4$

$a + 4b = -1$

$16a + \cancel{4b} = 44$

$\mp a \pm \cancel{4b} = \pm 1$

$15a = 45 \Rightarrow a = 3$

$a + 4b = -1$

$3 + 4b = -1$

$4b = -1 - 3$

$4b = -4 \Rightarrow b = -1$

3- عمليتي الجمع والضرب

$Z_1 + Z_2 = (a_1 + ib_1) + (a_2 + ib_2)$

$= (a_1 + a_2) + i(b_1 + b_2)$

$Z_1 * Z_2 = (a_1 + ib_1)(a_2 + ib_2)$

$= a_1 a_2 + i^2 b_1 b_2 + i a_1 b_2 + i a_2 b_1$

$= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$

$= a_1 a_2 - b_1 b_2 , a_1 b_2 + a_2 b_1$

4. عمليتي الطرح والقسمة حيث ان

$Z_1 - Z_2 = (a_1 + ib_1) - (a_2 + ib_2)$

$(a_1 - a_1) + i(b_1 - b_2)$

$Z = \frac{Z_1}{Z_2} \quad Z \neq 0$

$$Z = \frac{Z_1}{Z_2}$$

$$Z_1 = Z Z_2$$

$$a_1 + ib_1 = (a + ib)(a_2 + ib_2)$$

$$a_1 + ib_1 = (aa_2 - bb_2) + i(ab_2 + a_2 b)$$

$$a_1 = aa_2 - bb_2 \quad — ① \quad * \; a_2$$

$$b_1 = ab_2 + a_2 b \quad — ② \quad * \; b_2$$

$$a_1 a_2 = aa_2^2 - \cancel{bb_2 a_2} \quad ③$$

$$b_1 b_2 = ab_2^2 + \cancel{a_2 b_2 b} \quad ④$$

$$a_1 a_2 + b_1 b_2 = aa_2^2 + ab_2^2$$

$$a_1 a_2 + b_1 b_2 = a(a_2^2 + b_2^2)$$

$$a = \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}$$

H-W جد قيمة b

نضرب معادلة ① b_2

معادلة ② a_2

$$b = \frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}$$

$$Z = \frac{Z_1}{Z_2} = \frac{a_1 + ib_1}{a_2 + ib_2}$$

$$= \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i \frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}$$

$$\frac{1}{Z} = Z^{-1} = \frac{1}{a+ib} \cdot \frac{a-ib}{a-ib}$$ -5

$$= \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} + i \frac{-b}{a^2+b^2}$$

$$-Z = -(a+ib) = -a-ib$$ -6

Complex conjugate مرافق العدد

مرافق العدد $Z = a+ib$ هو $\bar{Z} = a-ib$

حيث يتم تغيير اشارة الجزء الخيالي من العدد العقدي

$$\bar{Z} = \overline{(a+ib)} = a-ib$$

خواص المرافق العقدي

① $z = a + ib \qquad z_1 = a_1 + ib_1$

$$z_2 = a_2 + ib_2$$

$$Re(z) = \frac{z + \bar{z}}{2} \Rightarrow z + \bar{z} = 2a = 2\,Re(z)$$

② $$z - \bar{z} = 2ib = 2i\,Im(z) \Rightarrow b = \frac{z - \bar{z}}{2i}$$

③ $$z \cdot \bar{z} = [Re(z)]^2 + [Im(z)]^2$$
$$= a^2 + b^2 = |z|$$

④ $\bar{\bar{z}} = z$

⑤ $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_1$

⑥ $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

⑦ $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2} \qquad z_2 \neq 0$

H.w prove that :-

Remark :- القوى

$i^2 = -1$

$i = \sqrt{-1}$

$i^3 = i^2 \cdot i = -1 * i = -i$

$i^4 = i^2 * i^2 = -1 * -1 = 1$

$i^5 = i^4 * i = 1 * i = i$

$i^7 = i^4 * i^2 \cdot i = 1 * -1 * i = -i$

قاعدة نقسم الاس على 4 والباقي هو الاس الجديد لـ i

$i^{25} = (i^4)^6 * i = i$

$i^{99} = (i^4)^{24} \cdot i^3 = i^3 = i^2 \cdot i = -i$

Absolute value القيمة المطلقة

اذا كانت a و b اعداد حقيقية فان العدد الحقيقي غير السالب $\sqrt{a^2+b^2}$ يسمى بالقيمة المطلقة للعدد المعقد $z = a+ib$ ويرمز له بالرمز $|z|$

$$|z| = |a+ib| = \sqrt{a^2+b^2}$$

حيث القيمة المطلقة للعدد المعقد z هي المسافة بين النقطة z ونقطة الاصل وكل عدد معقد يرافقه ثلاث اعداد حقيقية $[\,|z|,\ \mathrm{Re}\, z,\ \mathrm{Im}\, z\,]$

والعلاقات التي تربط هذه الاعداد :

$$|z|^2 = [\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2 = z \cdot \bar{z}$$

$$\because [\mathrm{Re}(z)]^2 \geq 0 \quad , \quad [\mathrm{Im}(z)]^2 \geq 0$$

$$|z|^2 \geq [\mathrm{Re}(z)]^2$$

$$|z| \geq |\mathrm{Re}\, z| \geq \mathrm{Re}(z)$$

$$|z| \geq |\mathrm{Im}(z)| \geq \mathrm{Im}(z)$$

properties of absolut value — خواص القيمة المطلقة -

1. $|z|^2 = z \cdot \bar{z}$

2. $|\bar{z}| = |z| = |-z|$

3. $|z_1 z_2| = |z_1|\,|z_2|$

4. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}, \quad z_2 \neq 0$

5. $|z_1 + z_2| \leqslant |z_1| + |z_2|$

6. $|z_1 - z_2| \geqslant |z_1| - |z_1|$

example: 1 Find the absolute value of the complex number $\dfrac{(4+3i)(1+i)}{7-i}$

sol:- $(4+3i)(1+i) = 1+7i$

$$\frac{1+7i}{7-i} \cdot \frac{7+i}{7+i} = \frac{50i}{49+1} = \frac{50i}{50} = i$$

$$\left|\frac{(4+3i)(1+i)}{7-i}\right| = |i| = \sqrt{1^2} = 1$$

example: 2 prove that $\overline{\bar{z} + 3i} = z - 3i$

sol:-

نفرض ان $z = a+ib$

$$\overline{\bar{z} + 3i} = \overline{\overline{a+ib} + 3i}$$

$$= \overline{a-ib+3i}$$

$$= \overline{a+i(3-b)}$$

$$= a-i(3-b)$$

$$= a-3i+bi$$

$$= a+ib-3i$$

$$= z-3i$$

$$\bar{\bar{z}} = z$$

$$\therefore \overline{\bar{z}+3i} = z-3i$$

example:- Find the complex number E if

$|E| = 1$, $Re(E^2) = 0$

Sol:- $E = a + ib$

$|E| = \sqrt{a^2 + b^2} = 1$

$a^2 + b^2 = 1 \Rightarrow 1)$

$E^2 = (a + ib)(a + ib) = a^2 - b^2 + 2iab$

$Rel = a^2 - b^2$

$Im = 2ab$

$Re(E^2) = a^2 - b^2 = 0 \Rightarrow 2)$

$\cancel{a^2} + b^2 = 1$

$\mp \cancel{a^2} \pm b^2 = 0$

$2b^2 = 1 \Rightarrow b^2 = \frac{1}{2} \Rightarrow b = \mp \frac{1}{\sqrt{2}}$

$2a^2 = 1 \Rightarrow a^2 = \frac{1}{2} \Rightarrow a = \mp \frac{1}{\sqrt{2}}$

$\therefore$ $\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

$\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

$-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

$-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

example:- write the number $\frac{3+2i}{1+i}$ in the form $a+ib$

$$\frac{3+2i}{1+i} = \frac{3+2i}{1+i} \cdot \frac{1-i}{1-i} = \frac{5-i}{1+1} = \frac{5-i}{2}$$

$$= \frac{5}{2} - \frac{1}{2}i$$

example:- Find the value of $\frac{z+2}{z+1}$ if $z = x+iy$

$$\frac{z+2}{z+1} = \frac{x+iy+2}{x+iy+1} = \frac{(x+2)+iy}{(x+1)+iy} \cdot \frac{(x+1)-iy}{(x+1)-iy}$$

$$= \frac{(x+2)(x+1) - (x+2)iy + (x+1)iy + y^2}{(x+1)^2+y^2}$$

$$= \frac{x^2+3x+2(-\cancel{x}-2+\cancel{x}+1)iy + y^2}{(x+1)^2+y^2}$$

$$= \frac{x^2+3x+2+y^2 - iy}{(x+1)^2+y^2}$$

$$= \frac{x^2+3x+2+y^2}{(x+1)^2+y^2} - i\frac{y}{(x+1)^2+y^2}$$

example:- Find the value $\dfrac{(3(i)^{30} - i^{19}}{2i - 1}$

$$\frac{3i^{30} - i^{19}}{2i-1} = \frac{3(i^{2})^{15} - (i^{2})^{9} * i}{-1+2i}$$

$$\frac{-3-(-1)i}{-1+2i} = \frac{(-3+i)}{-1+2i} * \frac{(-1-2i)}{-1-2i}$$

$$= \frac{3+6i-i-2i^{2}}{1+4}$$

$$= \frac{3+5i+2}{5} = \frac{5+5i}{5}$$

$$= \frac{5}{5} + \frac{5}{5}i = 1+i$$

H-w

① $\dfrac{4i^{9} - i^{14} + 8}{-4i+1}$, ② $\dfrac{5i^{20} + 2i^{7}}{3i-2}$

example:- what complex number equal

① square conjugate.

② Cube conjugate.

Sol:-

If $z = a + ib$

$$\bar{z} = \overline{a+ib} = a - ib$$

$$z = \bar{z}^2$$

$$a + ib = (a - ib)^2 = a^2 - b^2 - 2iab$$

$$a = a^2 - b^2 \quad \text{—}①$$

$$b = -2ab \quad \longrightarrow ②$$

من معادلة ② نجد a

$$a = \frac{b}{-2b} \Rightarrow a = \frac{-1}{2}$$ نعوض في ①

$$\frac{-1}{2} = \left(\frac{-1}{2}\right)^2 - b^2 \Rightarrow -\frac{1}{2} = \frac{1}{4} - b^2$$

$$\frac{-1}{2} - \frac{1}{4} = b^2$$

$$-b^2 = \frac{-2-1}{4} = \frac{-3}{4}$$

$$b = \mp\frac{\sqrt{3}}{2}$$

$$z = a + ib = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$= -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Geometric representation of polar Coordinates

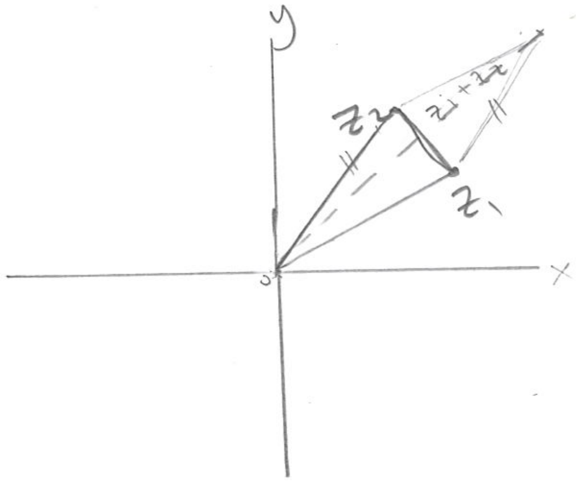
التمثيل الهندسي للاحداثيات القطبية.

- كل عدد عقدي $z = x + iy$ يمكن تمثيله بنقطة واحدة في المستوى (x, y) احداثياتها هي (x, y) وبالعكس
- كل نقطة في المستوي الاحداثي تقابل عدداً عقدياً واحداً
- يسمى المحور x بالمحور الحقيقي ، y بالمحور الخيالي ويسمى المستوي x, y بمستوى الاعداد العقدية .

$$z_1 = x_1 + iy_1$$

$$z_2 = x_2 + iy_2$$

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$



$$|z-1+3i| = (x-1)^2+(y+3)^2 = 4$$

$$(x-h)^2+(y-k)^2 = r^2$$

$$r^2 = 4$$

$$\boxed{r = 2}$$

$$(x-1)^2 = (x-h)^2$$

$$(x-1) = x-h$$

$$\cancel{x}-1-\cancel{x}+h=0$$

$$h=1$$

$$(y+3)^2 = (y-k)^2$$

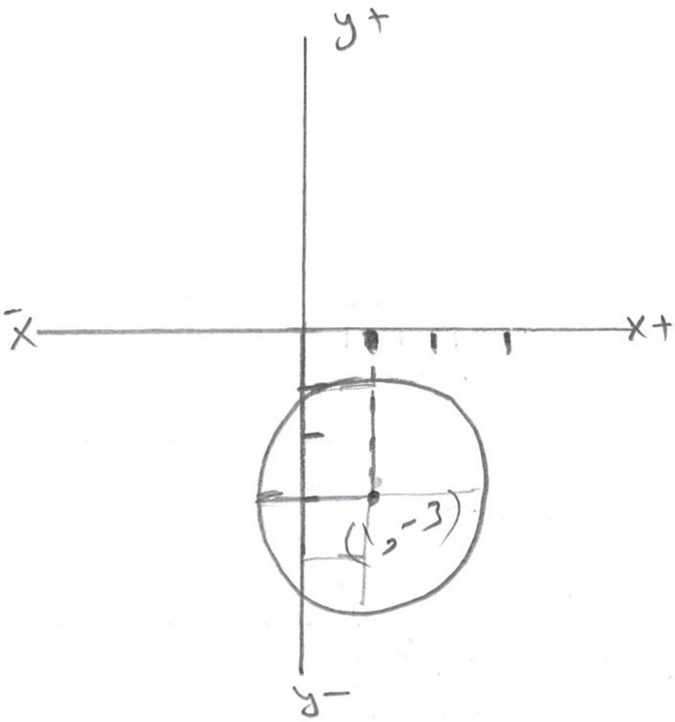
بالحيز

$$y+3 = y-k$$

$$\cancel{y}+3-\cancel{y}+k=0$$

$$k=-3$$

$\therefore$ $(1,-3)$ مركزها



Remark

1- الاعداد العقدية التي تقابل نقاط واقعة على محيط دائرة مركزها نقطة الاصل ونصف قطرها r حيث $r>0$ تحقق المعادلة التالية $|z| = r$

$$\sqrt{a^2+b^2} = r$$

$$a^2+b^2 = r^2$$

2- الاعداد العقدية الواقعة على محيط دائرة مركزها العدد العقدي $z_0 = x_0 + iy_0$ ونصف قطرها عدد حقيقي موجب مثل r تحقق المعادلة $|z - z_0| = r$

$$(x - x_0) + (y - y_0)i = r \Rightarrow \sqrt{(x-x_0)^2 + (y-y_0)^2} = r$$

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

example:- Geometrically represent the complex number that fulfills the equation :-

Sol:- $|z - 1 + 3i| = 2$

$z = x + iy = (x, y)$ نفرض

$$|z - 1 + 3i| = |(x+iy) - (1-3i)| = 2$$

$$|z - 1 + 3i| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = 2$$

Polar Coordinates for the complex numbers

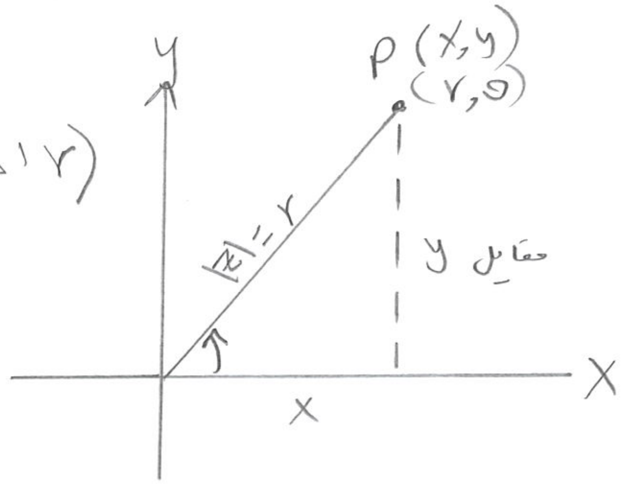
الاحداثيات القطبية للاعداد العقدية

لتكن (r, θ) احداثيات قطبية للنقطة P التي تقابل العدد العقدي على المستوي $z = x + iy$

$|z| = r$

$\cos\theta = \dfrac{x}{r}$ (r المقياس)

$\sin\theta = \dfrac{y}{r}$



$\therefore\ x = r\cos\theta$

$y = r\sin\theta$

$(x, y) = (r\cos\theta, r\sin\theta)$

$z = r(\cos\theta + i\sin\theta)$

* ان العدد الحقيقي r هو طول المتجه الذي يمثل z اي ان $|z| = r$

* والعدد الحقيقي θ يسمى بزاوية العدد العقدي z حيث تقاس θ بالتقدير الدائري وهي الزاوية التي يصنعها المتجه z مع المحور السيني الموجب باتجاه عكس عقارب الساعة.

$\therefore$ لكل عدد عقدي عدد لانهائي من القيم الحقيقية تختلف عن بعضها البعض بمضاعفات 2π.

* لكل $z \neq 0$ زاوية العدد العقدي الواقعة $(-\pi, \pi]$ تسمى بالزاوية الاساسية تمييزاً لها عن الزوايا الاخرى حيث تقع بين

$$-\pi < \operatorname{Arg} z \leq \pi$$

Arg z الزاوية الاساسية

$\theta = \arg z$

$$\arg z = \operatorname{Arg} z + 2K\pi$$

$$[K = 0, \mp 1, \mp 2, \mp 3 \ \ldots]$$

Remark

* لا يمكن تمثيل العدد $z = 0$ بالاحداثيات القطبية لان $\tan\theta = \frac{x}{y} = \frac{0}{0}$

Arg z The principle value القيمة الاساسية

arg z جميع كل Arg z

* لايجاد θ حسب العلاقة

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

example:- let z and w be two complex numbers then -

① $\arg(z \cdot w) = \arg z + \arg w$

let $z = |z| \left[\cos \theta_1 + i \sin \theta_1\right]$

$w = |w| \left[\cos \theta_2 + i \sin \theta_2\right]$

$$z \cdot w = |z| \cdot |w| \left[\underbrace{\cos\theta_1 \cos\theta_2} + i \cos\theta_1 \sin\theta_2 + i \sin\theta_1 \cos\theta_2 - \underbrace{\sin\theta_1 \sin\theta_2}\right]$$

$$= |z| \cdot |w| \left[\underbrace{\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2} + i\left[\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2\right]\right]$$

فك قوساً كلا $\rightarrow i$

$$= |z| \cdot |w| \left[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)\right]$$

$\therefore \arg(z \cdot w) = \arg z + \arg w$

H.W

② $\arg \dfrac{z}{w} = \arg z - \arg w$

③ $\arg(\overline{z}) = -\arg z = \arg \dfrac{1}{z}$

example :- Find $Arg(3)$

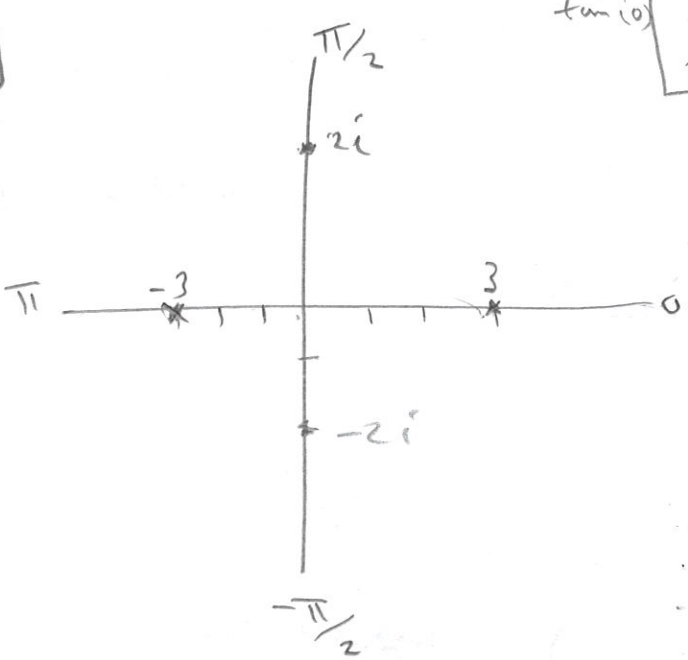
Sol:- $z = \overset{x}{3} + \overset{y}{0}i$

$$Arg(3) = \tan^{-1} \frac{y}{x}$$

$$Arg(3) = \tan^{-1} \frac{0}{3} = \tan^{-1}(0) = 0$$

$$\tan^{-1}(0) \begin{cases} 0 \\ \pi \\ 360 \\ 270 \end{cases}$$

$\therefore 0 \in (\pi, -\pi]$



example

Find $Arg(-3)$

$z = -3 + 0i$

$$Arg(-3) = \tan^{-1} \frac{0}{-3} = 0$$

$$Arg(-3) = \pi$$

example :- Find $Arg(2i) \Rightarrow z = 0 + 2i$

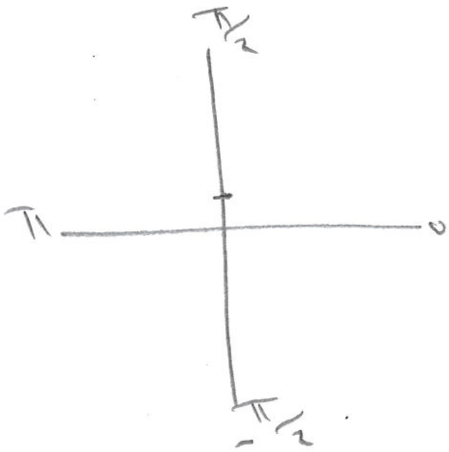
$$Arg(2i) = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{2}{0} = \frac{\pi}{2}$$

example $Arg(-2i)$

$z = 0 - 2i \Rightarrow Arg(-2i) = \tan^{-1}\left(\frac{-2}{0}\right) = -\frac{\pi}{2}$

غير معرف

example $\mathrm{Arg}\left(\frac{1+i}{1-i}\right)$



Sol:- $z = \frac{1+i}{1-i} * \frac{1+i}{1+i}$

$z = \frac{\cancel{2}i}{\cancel{2}} = 0 + i$ $\quad x=0 \quad y=1$

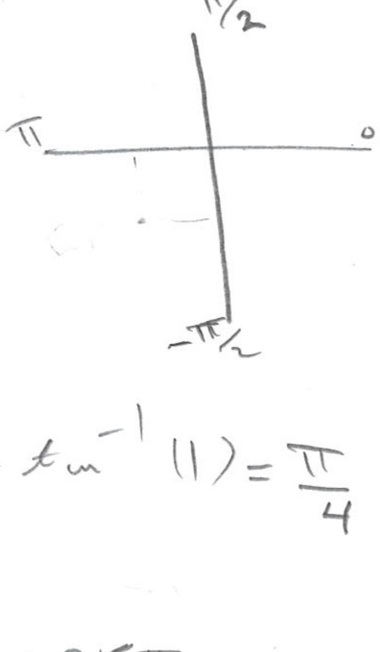
$\mathrm{Arg}\left[\frac{1+i}{1-i}\right] = \tan^{-1}\left(\frac{1}{0}\right) = \pi/2$

example

Find arg z if

① $z = \sqrt{3} + i$

$\mathrm{Arg}\, z = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6} = 30$



$\arg z = \mathrm{Arg}\, z + 2k\pi$

$= \frac{\pi}{6} + 2k\pi \qquad k \in \mathbb{Z}$

② $z = -1 - i$

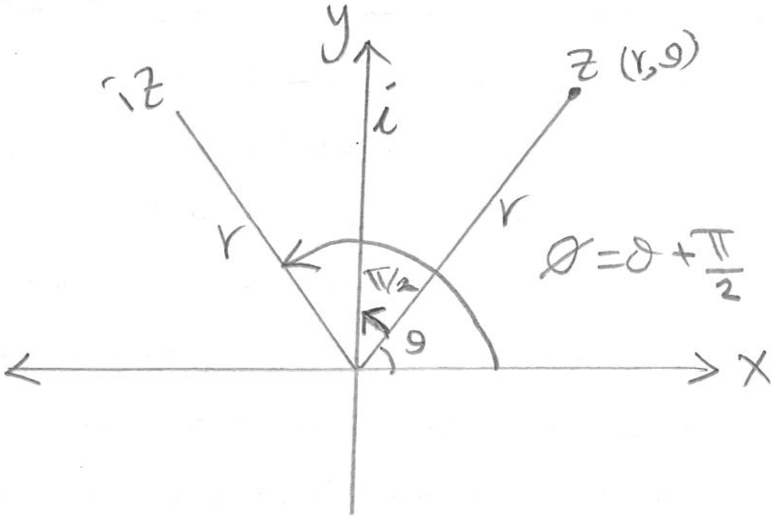
Sol:- $\mathrm{Arg} \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{-1}{-1} = \tan^{-1}(1) = \frac{\pi}{4}$

$\mathrm{Arg}\, z = \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$

$\arg z = \mathrm{Arg}\, z + 2k\pi = \frac{-3\pi}{4} + 2k\pi$

Properties of θ خواص الزاوية

١- حاصل ضرب اي عدد عقدي بالعدد i هو عبارة عن دوران المتجه الممثل للعدد العقدي بزاوية موجبة قدرها $\frac{\pi}{2}$



لو فرضنا ان $\boxed{z = r(\cos\theta + i\sin\theta)}$

$$i = z, \quad r = |z|$$

$$i = |z|(\cos\theta + i\sin\theta)$$

$$i = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$iz = r\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)(\cos\theta + i\sin\theta)$$

$$iz = r\left(\left(\cos\theta + \frac{\pi}{2}\right) + i\left(\sin\theta + \frac{\pi}{2}\right)\right)$$

$$\therefore \operatorname{Arg} iz = \theta + \frac{\pi}{2}$$

2- زاوية العدد المرافق يساوي زاوية العدد العقدي نفسه بعكس الاشارة

$$\theta = Arg\ Z$$

$$-\theta = Arg\ \bar{Z}$$

$$Z = r(\cos\theta + i\sin\theta$$

$$\bar{Z} = r(\cos\theta - i\sin\theta)$$

$$\bar{Z} = r((\cos(-\theta) + i\sin-\theta)$$

$$Arg\ \bar{Z} = -\theta = -Arg\ Z$$

3- θ ليست ذات قيمة وحيدة

$$Z = r(\cos\theta + i\sin\theta)$$

حيث r, θ هي الاحداثيات القطبية للعدد Z

$$\tan^{-1}\frac{y}{x},\quad x=0$$

$$\theta = \frac{\pi}{2} + 2K\pi \qquad K = 0, \mp 1, \mp 2 \ldots$$

غير معرفة $\theta = \quad x, 0 \quad y=0$

$$\therefore \theta = Arg\ Z$$

* ليست وحيدة بذلك يكون الصفر لانهائيات

4- اذا كان $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

① $z_1 \cdot z_2 = r_1 r_2\left[\cos\theta_1+\theta_2 + i(\sin\theta_1+\theta_2)\right]$

$\arg(z_1 * z_2) = \arg z_1 + \arg z_2$

② $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$

$\arg(z_1/z_2) = \arg z_1 - \arg z_2$

example:- Find Polar representation of

$\boxed{1}$ $z = 1 + \sqrt{3}\, i$

Sol:-

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (\sqrt{3})^2}$$

$$= \sqrt{1+3} = \sqrt{4} = 2$$

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{\sqrt{3}}{1} = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

$$\therefore z = r\,[\cos\theta + i\sin\theta]$$

$$z = 2\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]$$

H-W

① $z = 1 - i$

② $-2 + 2i$

③ $2\sqrt{3} - 2i$

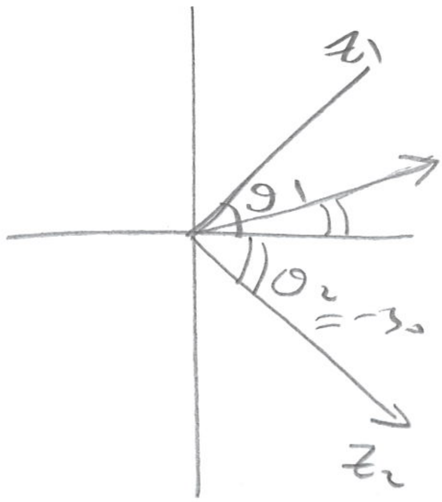
example :- Find $\arg z_1 \cdot z_2$

$z_1 = 1+i \quad z_2 = \sqrt{3} + i$

$\boxed{z_1 = 1+i}$

$r = \sqrt{1+1} = \sqrt{2}$

$z_1 = r_1 (\cos\theta_1 + i\sin\theta_1)$



$\sin\theta_1 = \frac{y_1}{r_1} = \frac{1}{\sqrt{2}}$

$\cos\theta_1 = \frac{x_1}{r_1} = \frac{1}{\sqrt{2}}$

$\therefore \theta_1 = \frac{\pi}{4} \Rightarrow \operatorname{Arg} z_1$

$z_2 = \sqrt{3} - i$

$r = |z_2| = \sqrt{3+1} = 2$

$$\left.\begin{aligned} \cos\theta_2 &= \frac{x_2}{r_2} = \frac{\sqrt{3}}{2} \\ \sin\theta_2 &= \frac{y_2}{r_2} = \frac{-1}{2} \end{aligned}\right\} \theta = \frac{-\pi}{6} = \operatorname{Arg} z_2$$

$\operatorname{Arg}(z_1 \cdot z_2) = \operatorname{Arg}\theta_1 + \operatorname{Arg}\theta_2$

$= \frac{\pi}{4} - \frac{\pi}{6} = 15^\circ$

$\arg z_1 = \operatorname{Arg} z_1 + 2n\pi$ $\qquad n = 0, \mp 1, \mp 2$

$\arg z_2 = \operatorname{Arg} z_2 + 2m\pi$ $\qquad m = 0, \mp 1, \mp 2$

$$\arg(z_1 \cdot z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$$

$$= \operatorname{Arg} z_1 + \operatorname{Arg} z_2 + 2(n+m)\pi$$

$\arg z_1 \cdot z_2 = 15^\circ + 2K\pi$ $\qquad \underline{K = n+m}$

example Find $\frac{z_1}{z_2}$ if

$z_1 = 8 + i8\sqrt{3}$

$z_2 = 2\sqrt{3} + 2i$

Sol:- $z_1 \Rightarrow r_1 = \sqrt{8^2 + (8^2) \cdot 3} = \sqrt{256} = 16$

$z_1 \Rightarrow r_2 = \sqrt{12 + 4} = 4$

$$\left.\begin{array}{l} \cos\theta_1 = \frac{x}{r} = \frac{8}{16} = \frac{1}{2} \\ \sin\theta_1 = \frac{y}{r} = \frac{8\sqrt{3}}{16} = \frac{\sqrt{3}}{2} \end{array}\right\} \Rightarrow \theta = \frac{\pi}{3} = 60^\circ$$

$$\left.\begin{array}{l} \cos\theta_2 = \frac{x}{r} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2} \\ \sin\theta_2 = \frac{y}{r} = \frac{2}{4} = \frac{1}{2} \end{array}\right\} \Rightarrow \theta = \frac{\pi}{6} = 30^\circ$$

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{16}{4} \frac{\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]}{\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]}$$

$$= 4\left[\cos\left[\frac{\pi}{3} - \frac{\pi}{6}\right] + i\sin\left(\frac{\pi}{3} - \frac{\pi}{6}\right)\right]$$

$$= 4\left[\cos\theta\ \frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$$

example :- if $z = 4 - 4\sqrt{3}i$ write z in polor formula and find argument z.

Sol :-
$$z = 4 - 4\sqrt{3}i$$

$$r = |z| = \sqrt{a^2 + b^2} = \sqrt{16 + 48} = \sqrt{64} = 8$$

$$\left.\begin{aligned} \cos\theta &= \frac{a}{r} = \frac{4}{8} = \frac{1}{2} \\ \sin\theta &= \frac{b}{r} = \frac{4\sqrt{3}}{8} = \frac{\sqrt{3}}{2} \end{aligned}\right\} \theta = \frac{\pi}{3}$$

b is negative

$$\arg z = 2\pi - \theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$$

$$z = = 4 - 4\sqrt{3}i = 8\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$

Remark :-

If a is negative $\Rightarrow \arg z = \pi - \theta$

If b is negative $\Rightarrow \arg z = 2\pi - \theta$

if a, b negative $\Rightarrow \arg z = \frac{3\pi}{2} - \theta$

example :- $\left(\frac{5i}{2+i}\right)$ write z in polar formula.

$\left(\frac{z}{w}\right)$

$z = 0 + 5i$

$r = |z| = \sqrt{25} = 5$

$\cos\theta_1 = \frac{a}{r_1} = \frac{0}{5} = 0$

$\sin\theta_1 = \frac{b}{r_1} = \frac{5}{5} = 1$

$\theta = \frac{\pi}{2}$

$z = 5\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$w = 2 + i$

$r_2 = |w| = \sqrt{4+1} = \sqrt{5}$

$\cos\theta_2 = \frac{2}{\sqrt{5}}$

$\sin\theta_2 = \frac{1}{\sqrt{5}}$

$$\frac{z}{w} = \frac{5\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}}{\sqrt{5}(\cos\theta_2 + i\sin\theta_2)} \cdot \frac{\sqrt{5}((\cos\theta_2 - i\sin\theta_2)}{\sqrt{5}((\cos\theta_2 - i\sin\theta_2)}$$

$$\frac{z}{w} = \frac{\cancel{5}\sqrt{5}}{\cancel{5}} \frac{\cos\frac{\pi}{2}\cos\theta_2 - \cos\frac{\pi}{2} i\sin\theta_2 + i\sin\frac{\pi}{2}\cos\theta_2 + \sin\frac{\pi}{2}\sin\theta_2}{(\cos\theta_2^2 + \sin^2\theta_2)}$$

$$= \sqrt{5}\ \underbrace{\cos\frac{\pi}{2}\cos\theta_2 + \sin\frac{\pi}{2}\sin\theta_2}\ + i\left(\sin\frac{\pi}{2}\cos\theta_2\right)$$

$$\frac{z}{w} = \sqrt{5}\left[\left(\cos\frac{\pi}{2} - \theta_2\right) + i\left(\sin\frac{\pi}{2} - \theta_2\right)\right]$$

$$= \sqrt{5}\left(\cos\theta_2 + i\sin\theta_2\right)$$

$$= \sqrt{5}\left(\frac{1}{\sqrt{5}} + \frac{2}{\sqrt{5}}i\right)$$

$$= \frac{\cancel{\sqrt{5}}}{\cancel{\sqrt{5}}} + \frac{2\cancel{\sqrt{5}}}{\cancel{\sqrt{5}}}i = \boxed{1+2i}$$

prov that

(a) z is real part $\Longleftrightarrow$ $\bar{z} = z$

(b) z is real or imaginary iff $(\bar{z})^2 = z^2$

Prove :-

(a) if z is real

$$z = a + 0i \Rightarrow \bar{z} = a$$

$$\therefore \bar{z} = z$$

$$a + ib = a - ib$$

$$\cancel{a} + ib \cancel{-a} + ib = 0 \Rightarrow 2ib = 0 \Rightarrow 2i \neq 0$$

$$b = 0$$

$$\therefore z = a + 0i = a$$

z is real part

(b) if z is real or imaginary $(\bar{z})^2 = z^2$

(1) if z is real

$z = a + i0$

$\because \bar{z} = z \Rightarrow (\bar{z})^2 = z^2$

(2) if z Imaginary

$z = ib \Rightarrow \bar{z} = -ib$

$z^2 = (ib)^2 = -b^2$

$(\bar{z})^2 = (-ib)^2 = -b^2$

$\therefore z^2 = (\bar{z})^2$

$(a - ib)^2 = (a + ib)^2$

$a^2 - 2iab \; b^2 = a^2 + 2iab - b^2$

$\cancel{a^2} - 2iab \cancel{- b^2} \cancel{- a^2} - 2iab \cancel{+ b^2} = 0$

$-4iab = 0$

$-4i \neq 0$

$\therefore ab = 0 \quad$ if $a = 0 \Rightarrow z = -ib$

or $b = 0 \Rightarrow z = a$.

Example:- if $x \in C$, $\bar{x}$ conjugated to x

Find the complex number that fulfills the $3x + \bar{x} = 2i + 3$.

Sol:- $x = a + ib$ $\bar{x} = a - ib$

$$3(x) + \bar{x} = 2i + 3$$

$$3(a + ib) + (a - ib) = 2i + 3$$

$$3a + 3ib + a - ib = 3i + 3$$

$$3a + a = 3 \Rightarrow 4a = 3 \Rightarrow a = \frac{3}{4}$$

$$3b - b = 2 \Rightarrow \Rightarrow 2b = 2 \Rightarrow b = 1$$

$$x = a + ib$$

$$\boxed{x = \frac{3}{4} + i}$$

example :- If $L = \frac{7-i}{2-i}$, $K = \frac{13-i}{4+i}$

Prove the L, K are conjugated

$L^2 K = L K^2$.

sol :-

$$K = \frac{13-i}{4+i} = \frac{13-i}{4+i} * \frac{4-i}{4-i}$$

$$= \frac{52 - 13i - 4i + i^2}{4^2 + 1^2} = \frac{51 - 17i}{17}$$

$$= \frac{51}{17} - \frac{17}{17} i = 3 - i$$

$$L = \frac{7-i}{2-i} = \frac{7-i}{2-i} * \frac{2+i}{2+i}$$

$$= \frac{14 + 7i - 2i - i^2}{4+1}$$

$$= \frac{15 + 5i}{5} = 3 + i$$

$$(3+i) + (3-i) = 6 \in R$$

$$(3+i)(3-i) = 3^2 + 1^2 = 10 \in R$$

$$L^2 K + L K^2 = LK(L+K) = 10(6) = 60$$

$L . K$ conjugated.

A gebric properties of the complex number

1- $z_1+z_2=z_2+z_1$

$z_1 . z_2 = z_2 \cdot z_1$

2- $(z_1+z_2)+z_3 = z_1+(z_2+z_3)$

3) $z_1(z_2+z_3) = z_1 z_2 + z_1 z_3$

examples— prov $z \cdot z^{-1} = 1$

$z = a+ib$

$$z^{-1} = \frac{1}{a+ib} = \frac{1}{a+ib} \cdot \frac{a-ib}{a-ib}$$

$$= \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{ib}{a^2+b^2}$$

$$z \cdot z^{-1} = a+ib \cdot \frac{a-ib}{a^2+b^2}$$

$$= \frac{a^2+b^2}{a^2+b^2} = 1$$

example:- Prove that

$$|(2\bar{z}+5)(\sqrt{2}-i)| = \sqrt{3}\,|2z+5|$$

$$z = a+ib \qquad \bar{z} = a-ib$$

$$|2(a-ib)+5)(\sqrt{2}-i)| =$$

$$|(2a-2ib+5)(\sqrt{2}-i)| =$$

$$|2a\sqrt{2} - 2\sqrt{2}ib + 5\sqrt{2} - i2a + 2i^2b - 5i| =$$

$$|2a\sqrt{2} - 2\sqrt{2}ib + 5\sqrt{2} - 2ai - 2b - 5i| =$$

$$|(2a\sqrt{2} + 5\sqrt{2} - 2b) + i(2\sqrt{2}b + 2a + 5)| =$$

$$|(\sqrt{2}(2a+5) - 2b) - i((2a+5) + (2\sqrt{2}b))| =$$

$$\sqrt{\sqrt{2}(2a+5) - 2b)^2 - ((2a+5) + (2\sqrt{2}b))^2} =$$

$$\sqrt{(2(2a+5)^2 - 4\sqrt{2}\cancel{(2a+5)b} + 4b^2) + (2a+5)^2 + 4\sqrt{2}\cancel{b(2a+5)} + 8b^2}$$

$$= \sqrt{3(2a+5)^2 + 12b^2}$$

$$= \sqrt{3((2a+5)^2 + 4b^2)} = \sqrt{3}\sqrt{(2a+5)^2 + 4b^2}$$

$$= \sqrt{3}\sqrt{(2a+5)^2 + (2b)^2} = \sqrt{3}\,|(2a+5) + i2b|$$

$$= \sqrt{3}\,|(2a+2ib)+5| = \sqrt{3}\,|2(a+ib)+5|$$

$$= \sqrt{3}\,|2z+5| .$$

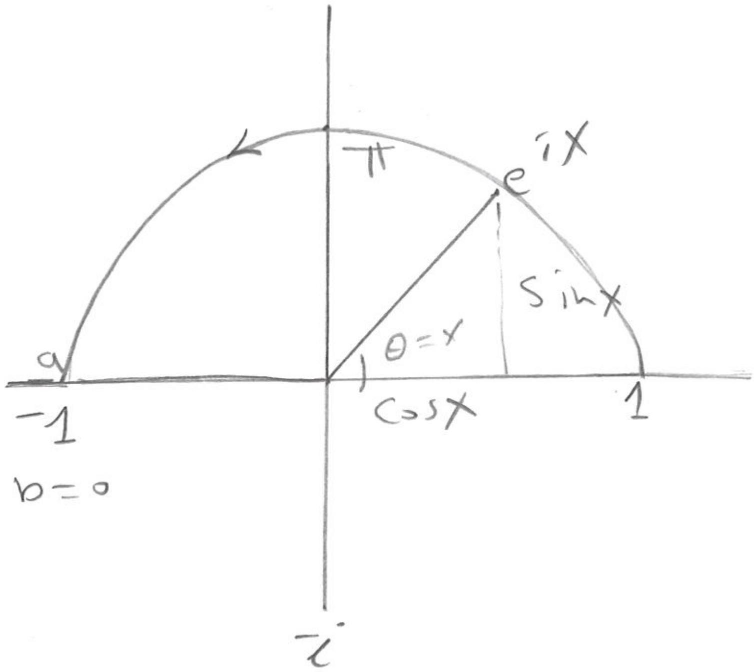
Euler's identity

متطابقة اويلر

$$e^{\pi i} + 1 = 0$$

or

$$re^{i\theta} = a + ib$$



$$(1)\ e^{i\pi} = (-1) + (0)i$$

$$e^{i\pi} = -1$$

$$e^{i\pi} + 1 = 0$$

$$e^{i\pi} = \cos\pi + i\sin\pi$$

$$e^{i\pi} = -1 + 0i$$

$$e^{i\pi} = -1 \Rightarrow \boxed{e^{i\pi} + 1 = 0}$$

prove that $e^{i\theta} = \cos\theta + i\sin\theta$

sol:-

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) \qquad (1)$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \frac{1}{i}\int \frac{i\,dx}{\sqrt{1+x^2}}$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \frac{1}{i}\ln\left(\sqrt{1-x^2}+ix\right) \qquad (2)$$

$$\sin^{-1}(x) = \frac{1}{i}\ln\left(\sqrt{1-x^2}) + ix\right)$$

$$i\,\underbrace{\sin^{-1}(x)}_{\theta} = \ln\left(\underbrace{\sqrt{1-x^2}}_{\cos\theta} + i\underbrace{x}_{\sin\theta}\right)$$

if $\sin^{-1}x = \theta \Rightarrow x = \sin\theta$

$$\sqrt{1-x^2} = \cos\theta$$

$$i\theta = \ln\left[\cos\theta + i\sin\theta\right]$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

$$Z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

- يمكن كتابة العدد العقدي
حسب صيغة متسلسلة تايلر

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} - \frac{i\theta^5}{5!}$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} - \frac{\theta^5}{5!} \cdots\right)$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Remark 1

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Euler's Formula صيغة اويلر

Using the symbol $e^{i\theta}$ or $\exp(i\theta)$ is defined by Euler's Formula for any real values of θ

as $e^{i\theta} = \cos\theta + i\sin\theta$

$$\boxed{Z = r(\cos\theta + i\sin\theta) = re^{i\theta}}$$

example :- if $Z = -1$ Find argument Z and Euler's Formula.

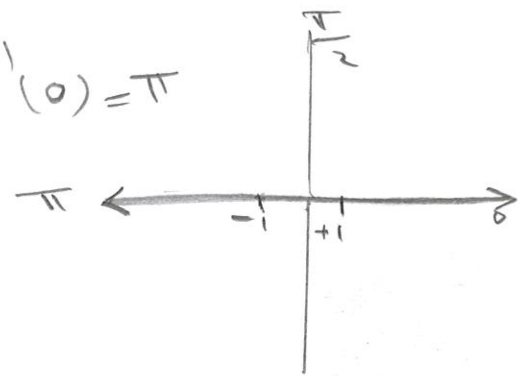
Sol :- $\arg Z = \operatorname{Arg} Z + 2K\pi \qquad K \in \mathbb{Z}$

$\operatorname{Arg} Z = \tan^{-1}\frac{y}{x}$

$Z = -1 + 0i \Rightarrow x = -1 \quad y = 0$

$\operatorname{Arg} Z = \tan^{-1}\left(\frac{0}{-1}\right) = \tan^{-1}(0) = \pi$

$\arg Z = \pi + 2K\pi$



$r = |Z| = |-1| = 1$

$\theta = \pi$

$Z = re^{i\theta} = 1e^{\pi i}$

example :- Find Euler's Formula :-

صيغة أويلر

$\boxed{1}$ $Z = 1 + i$

$$r = \sqrt{x^2 + y^2} = \sqrt{1+1} = \sqrt{2}$$

$$\theta = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{1}{1} = \tan^{-1} 1 = \frac{\pi}{4}$$

$$Z = r e^{i\theta}$$

$$Z = \sqrt{2}\, e^{\frac{\pi}{4} i}$$

$b = b+$

$\boxed{2}$ $z = \sqrt{3} - i$

$$r = \sqrt{(\sqrt{3})^2 + (1)^2} = \sqrt{4} = 2$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$$

$$= -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$$

$$Z = r e^{i\theta}$$

$$Z = 2\, e^{-\frac{\pi}{6} i}$$

$\boxed{3}$ $Z = -1 - i$

$b = -$

$$r = \sqrt{1+1} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1} \frac{-1}{-1} = \tan^{-1} 1 = \frac{\pi}{4}$$

$-1-i$ في الربع الثالث

$$\theta = \frac{\pi}{4} - \pi = \frac{-3\pi}{4}$$

$$Z = r e^{i\theta} = \sqrt{2}\, e^{-\frac{3}{4}\pi i}$$

Example:- Express z in the form $x+yi$

[1] $z = 2e^{-\frac{\pi}{4}i}$

$\frac{\pi}{4} = 45$

$z = r\left[\cos\theta + i\sin\theta\right]$

$z = 2\left[\cos -\frac{\pi}{4} + i\sin -\frac{\pi}{4}\right]$

$z = 2\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right]$

$z = 2\left[\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right]$

$= \frac{\sqrt{2}\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}} - i\frac{\sqrt{2}\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}} = \sqrt{2} - i\sqrt{2}$

$\boxed{z = \sqrt{2} - \sqrt{2}i}$

[2] $z = 3e^{\frac{\pi}{3}i}$

$z = 3\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]$

$\frac{\pi}{3} = 60$

$= 3\left[\frac{1}{2} + i\frac{\sqrt{3}}{2}\right]$

$\boxed{z = \frac{3}{2} + \frac{3\sqrt{3}}{2}i}$

example

[3] $Z = -5e^{\frac{5\pi}{6}i}$

$$Z = -5\left[\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right]$$

$$= -5\left[-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right]$$

$$\boxed{Z = \frac{5\sqrt{3}}{2} + \frac{1}{2}i}$$

---

[4] show that $|e^{i\theta}| = 1$

sol:- $Z = r[\cos\theta + i\sin\theta]$

$$Z = [\cos\theta + i\sin\theta] = e^{i\theta}$$

$$|e^{i\theta}| = |\cos\theta + i\sin\theta|$$

$$|e^{i\theta}| = \sqrt{\cos\theta^2 + \sin\theta^2} = \sqrt{1} = 1$$

---

[5] Find $|e^{\frac{\pi}{2}i}|$

$$|e^{\pi/2 i}| = \left|\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right|$$

$$= |0 + i(1)| = |i|$$

$$= \sqrt{0+1} = \sqrt{1} = 1$$

خواص صيغة اويلر

Products and quatients in exponential form

Let $z_1 = r_1 e^{i\theta_1}$ $\quad z_2 = r_2 e^{i\theta_2}$ Then

$\boxed{1}$ $z_1 \cdot z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$

$\boxed{2}$ $\frac{1}{z} = \frac{1}{r} e^{-i\theta}$ $\qquad$ $\boxed{3}$ $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$

$\boxed{ex}$ prove that (1)

$z_1 \cdot z_2 =$

$z_1 = r_1 e^{i\theta_1} = r_1 [\cos\theta_1 + \sin\theta_1]$

$z_2 = r_2 e^{i\theta_2} = r_2 [\cos\theta_2 + i\sin\theta_2]$

$$z_1 \cdot z_2 = r_1 r_2 [\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 + i(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2)]$$

$$= r_1 r_2 [\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 + i[\cos\theta_1 \sin\theta_2 + \sin\theta_2 \cos\theta_2]$$

$$= r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

$$z_1 \cdot z_2 = r_1 r_2 e^{i[\theta_1+\theta_2]}$$

Ex:- Let $z = 1-i$ and $z_2 = 1+\sqrt{3}i$ Find using Euler's formula:-

$\boxed{1}$ $z_1 . z_2$

Sol:- $z_1 = 1-i$

$r = \sqrt{1+1} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{-1}{1}\right) = \tan^{-1}(-1)$

$\therefore \theta = \frac{-\pi}{4}$

$z_1 = \sqrt{2}\, e^{-\frac{\pi}{4} i}$

$z_2 = 1+\sqrt{3}\, i$

$r = \sqrt{1+3} = 2$

$\theta = \tan^{-1}\frac{\sqrt{3}}{1} = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

$z_2 = 2\, e^{i\frac{\pi}{3}}$

① $z_1 . z_2 = r_1 r_2 \, e^{i(\theta_1+\theta_2)} = 2\sqrt{2}\, e^{i\left(\frac{-\pi}{4}+\frac{\pi}{3}\right)}$

$z_1 . z_2 = 2\sqrt{2}\, e^{i\left(\frac{\pi}{12}\right)}$

② H.w $\frac{z_1}{z_2}$

3) H.w $\frac{1}{z_1}$

ex :- writ the numbers the polar formula :-

$$Z = 2 + i2\sqrt{3}$$

Sol :- $Z = 2 + i2\sqrt{3}$

$$r = \sqrt{4 + 4 \cdot 3} = \sqrt{16} = 4$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \Rightarrow \theta = \tan^{-1} \frac{2\sqrt{3}}{2} = \frac{\pi}{3}$$

$$Z = r(\cos\theta + i\sin\theta)$$

$$Z = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$= 4e^{\frac{i\pi}{3}}$$

---

ex :- $Z = -5 + 5i$

Sol : $r = \sqrt{25 + 25} = 5\sqrt{2}$

$$\theta = \tan^{-1}\frac{y}{x} = \frac{5}{-5} = \frac{-\pi}{4}$$

$$Z = r\left[\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right] = 5\sqrt{2}\left(\cos\left(\frac{-\pi}{4}\right) + i\sin\frac{-\pi}{4}\right)$$

$$Z = 5\sqrt{2}\, e^{\frac{-\pi}{4}i}$$

ex show that $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$ using Euler's formula.

Sol:-

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\mp \quad e^{-i\theta} = \mp\cos\theta \pm i\sin\theta$$

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta \qquad \text{بالجمع } ①$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \text{بالطرح } ②$$

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$$

$$\therefore \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\sin^3\theta = \left[\frac{e^{i\theta} - e^{-i\theta}}{2i}\right]^3 = \frac{\left(e^{i\theta} - e^{-i\theta}\right)^3}{2^3 i^3}$$

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

$$\therefore = \frac{e^{3i\theta} - 3e^{2i\theta} \cdot e^{-i\theta} + 3e^{i\theta} e^{-2i\theta} - e^{-3i\theta}}{-8i}$$

$$\sin^3\theta = \frac{e^{3i\theta - 3i\theta}}{-8i} + \frac{3e^{i\theta} - 3e^{-i\theta}}{+8i}$$

$$= \frac{-1}{4}\sin 3\theta + \frac{3}{4}\sin\theta$$

$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$$

De Moivre's formula نظرية دي-موفر

if $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$z_1 \cdot z_2 = r_1 r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$$

$$z_1 \cdot z_2 \cdots z_n = r_1 r_2 \cdots r_n\left[\cos(\theta_1+\theta_2+\cdots\theta_n) + i\sin(\theta_1+\theta_2 \cdots \theta_n)\right]$$

$$z_1 = z_2 = z_n$$

$$z^n = r^n\left[\cos(n\theta) + i\sin(n\theta)\right]$$

$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

$$\cancel{r^n}(\cos\theta + i\sin\theta)^n = \cancel{r^n}(\cos(n\theta) + i\sin(n\theta))$$

$$\boxed{\cos\theta + i\sin\theta = \left(e^{i\theta}\right)^n = e^{in\theta} = \cos n\theta + i\sin n\theta}$$

نظرية دي موفر بصيغة اويلر

<u>ex</u> write $(\sqrt{3}+i)^7$ in the form $x+yi$

Sol:-

$$z = \sqrt{3} + i$$

$$r = \sqrt{x^2+y^2} = \sqrt{3+1} = \sqrt{4} = 2$$

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\theta = \frac{\pi}{6}$$

$$z = r(\cos\theta + i\sin\theta)$$

$$z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

$$z^7 = 2^7\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$$

$$= 128\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$$

$$z^7 = -64\sqrt{3} - 64i$$

$\frac{7\pi}{6} = 30 \times 7$
$= 210$
في الربع الثالث
$\cos^{-}$ $\sin^{-}$

ex write $z = \left[\frac{-1+\sqrt{3}i}{\sqrt{3}-i}\right]^5$ in the form $x+yi$

sol:-

$$z = \frac{-1+\sqrt{3}i}{\sqrt{3}-i} * \frac{\sqrt{3}+i}{\sqrt{3}+i}$$

$$= \frac{-\sqrt{3} - i + 3i - \sqrt{3}}{3+1}$$

$$= \frac{(-\sqrt{3}-\sqrt{3}) + (-1+3)i}{4}$$

$$z = \frac{-2\sqrt{3}+2i}{4} = \frac{-\sqrt{3}}{2} + \frac{i}{2}$$

$$r = \sqrt{\left(\frac{-\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}+\frac{1}{4}} = 1$$

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{\frac{1}{\cancel{2}}}{\frac{-\sqrt{3}}{\cancel{2}}}\right) = \tan^{-1}\frac{-1}{\sqrt{3}} = -\tan^{-1}\frac{1}{\sqrt{3}}$$

$$\theta = \frac{-\pi}{6}$$

$z = r(\cos\theta + i\sin\theta$

$z = \cos\frac{-\pi}{6} + i\sin\frac{-\pi}{6}$

$z = \left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$

$z^5 = \cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}$

$= -\frac{\sqrt{3}}{2} - \frac{1}{2}i$

ex) prove that Demoivrés formula.

$\boxed{1}$ $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$(\cos\theta + i\sin\theta)^2 = \cos^2\theta + 2i\sin\theta\cos\theta + i^2\sin^2\theta$$

$$(\cos 2\theta + i\sin 2\theta) = \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta$$

$$\boxed{\cos 2\theta = \cos^2\theta - \sin^2\theta}$$

$$\boxed{\sin 2\theta = 2\sin\theta\cos\theta}$$

$\boxed{2}$ $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$$(\cos\theta + i\sin\theta)^3 =$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(\cos\theta + i\sin\theta)^3 = \cos\theta^3 + 3\cos^2\theta\, i\sin\theta + 3\cos\theta\sin\theta^2 i^2 + i^3\sin\theta^3$$

$$(\cos 3\theta + i\sin 3\theta) = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$

$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ &= \cos^3\theta - 3(1-\cos^2\theta)\cos\theta \\ &= \cos^3\theta - 3\cos\theta + 3\cos^3\theta \\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

$$\sin 3\theta = 3\cos^2\theta \sin\theta - \sin^3\theta$$

$$= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta$$

$$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta$$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

ex) Find $\arg (1-i\sqrt{3})^3$

Sol:-

$z = 1-i\sqrt{3}$

$r = \sqrt{1+3} = \sqrt{4} = 2$

$z = r(\cos\theta + i\sin\theta)$

$1-i\sqrt{3} = 2(\cos\theta + i\sin\theta)$

$\frac{1}{2} - i\frac{\sqrt{3}}{2} = \cos\theta + i\sin\theta$

$\cos\theta = \frac{1}{2}$, $\sin\theta = \frac{-\sqrt{3}}{2}$ $\Rightarrow \theta = \frac{-\pi}{3}$

$\theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

$z^n = r^n(\cos n\theta + i\sin n\theta)$

$(1-i\sqrt{3})^3 = 2^3\left(\cos 3\cdot\frac{5\pi}{3} + i\sin 3\cdot\frac{5\pi}{3}\right)$

$= 2^3(\cos 5\pi + i\sin 5\pi)$

$\arg(1-i\sqrt{3})^3 = \operatorname{Arg}(1-i\sqrt{3})^3 + 2\pi k$

$\arg(1-i\sqrt{3})^3 = 5\pi + 2\pi k$

ex Find the polar $(2\sqrt{3}+2i)(3-3\sqrt{3}i)$

Sol:-

$z = 2\sqrt{3}+2i$

$r = \sqrt{4 \cdot 3+4} = \sqrt{16} = 4$

$z = r(\cos\theta + i\sin\theta)$

$2\sqrt{3}+2i = 4(\cos\theta + i s\theta)$

$\frac{2\sqrt{3}}{4} + \frac{2}{4} i = \cos\theta + i\sin\theta$

$\frac{\sqrt{3}}{2} + \frac{1}{2} i = \cos\theta + i\sin\theta \Rightarrow \sin\theta = \frac{1}{2}$

$\cos\theta = \frac{\sqrt{3}}{2}$

$\theta = \frac{\pi}{6}$

$z = 3-3\sqrt{3}i$

$r = \sqrt{9+9 \cdot 3} = \sqrt{36} = 6$

$3-3\sqrt{3}i = 6(\cos\theta + i\sin\theta)$

$\frac{3}{6} - \frac{3}{6}\sqrt{3}i = \cos\theta + i\sin\theta$

$\frac{1}{2} - \frac{\sqrt{3}}{2} i = \cos\theta + i\sin\theta$

$\cos\theta = \frac{1}{2}, \sin\theta = -\frac{\sqrt{3}}{2} \Rightarrow \theta = -\frac{\pi}{3} = 2\pi - \frac{\pi}{3}$

$\theta = \frac{5\pi}{3}$

$$(2\sqrt{3}+2i)(3-3\sqrt{3}i)=4*6\left(\cos\left(\frac{\pi}{6}+\frac{5\pi}{3}\right)+i\sin\left(\frac{\pi}{6}+\frac{5\pi}{3}\right)\right)$$

$$=24\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)$$

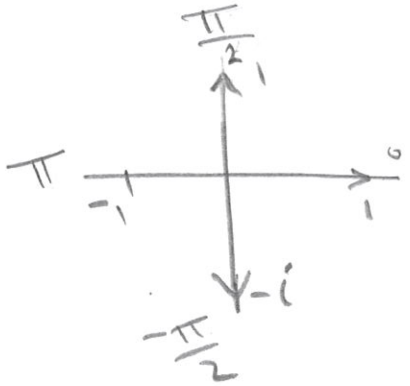
ex) Find the angle for the $i, -i, 1, -1$

sol:- $z = x+iy$

$z = r(\cos\theta + i\sin\theta)$

$z = 1 \Rightarrow r = 1$

$$\left.\begin{aligned}&1 = \cos\theta + i\sin\theta\\ &\cos\theta = 1 \quad \sin\theta = 0\end{aligned}\right\}\boxed{\theta = 0}$$



$z = -1$

$-1 = \cos\theta + i\sin\theta$

$$\left.\begin{aligned}&\cos\theta = -1\\ &\sin\theta = 0\end{aligned}\right\}\boxed{\theta = \pi}$$

$z = i$

$0 + i = \cos\theta + i\sin\theta$

$$\left.\begin{aligned}&\cos\theta = 0\\ &\sin\theta = 1\end{aligned}\right\}\boxed{\theta = \frac{\pi}{2}}$$

$z = -i$

$0 - i = \cos\theta + i\sin\theta \Rightarrow \begin{aligned}&\cos\theta = 0\\ &\sin\theta = -1\end{aligned} \Rightarrow \theta = \frac{-\pi}{2}$

ex $(-1+i)^7 = -8(1+i)$ prove that).

$z = -1 + i$

$|z| = r = \sqrt{2}$

$z = r \cos\theta + i\sin\theta$

$-1 + i = \sqrt{2}(\cos\theta + i\sin\theta)$

$\frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i = \cos\theta + i\sin\theta$

$\cos\theta = \frac{-1}{\sqrt{2}}$

$\sin\theta = \frac{1}{\sqrt{2}}$ $\Rightarrow \theta = \frac{\pi}{4}$

$\theta = \pi - \frac{\pi}{4} \Rightarrow \theta = \frac{3\pi}{4}$

$z^n = r^n(\cos n\theta + i \sin n\theta)$

$z^7 = (\sqrt{2})^7 = \cos 7 \cdot \frac{3\pi}{4} + i\sin 7 \cdot \frac{3\pi}{4}$

$z^7 = 8\sqrt{2}\left(\cos \frac{21\pi}{4} + i \sin \frac{21\pi}{4}\right)$

$= 8\sqrt{2}\left(\cos\left(4\pi + \frac{5\pi}{4}\right) + i\sin\theta\left(4\pi + \frac{5\pi}{4}\right)\right)$

$= 8\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

$= 8\sqrt{2}\left(\cos \pi + \frac{\pi}{4} + i\sin \pi + \frac{\pi}{4}\right)$

$= 8\sqrt{2}\left(-\cos\frac{\pi}{8} - i\sin\frac{\pi}{4}\right) = 8\sqrt{2}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$

$= -\frac{8\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}} - \frac{8\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}} i$

$= -8(1 + i)$